Supplemental material (File S2) for *Fixation dynamics of beneficial alleles in prokaryotic polyploid chromosomes and plasmids* (Santer et al., bioRxiv, 2021)

In[∘]:=
```
ClearAll["Global`*"]
$Assumptions = {_ ∈ Reals, n > 1}
```

Out[∘]= $\{\_ \in \mathbb{R}, n > 1\}$

Solution of $x_{\mathrm{mut}}(t)$

In[∘]:=
```
sol = DSolve[x'[t] == -s * x[t]^2 + x[t] (s + a * (b * Exp[b * t])/(a * Exp[b * t] + (b - a))) && x[0] == f, x, t]
```

Out[∘]= $\left\{\left\{x \to \text{Function}\left[\{t\}, \frac{e^{s t}\left(-a+b+a e^{b t}\right) f (b+s)}{b^2+a b f-b^2 f-a b e^{s t} f+b^2 e^{s t} f+b s-b f s-a e^{s t} f s+b e^{s t} f s+a e^{(b+s) t} f s}\right]\right\}\right\}$

(There is only a single solution (as expected).)

In[∘]:=
```
xmut[t_] = FullSimplify[x[t] /. sol[[1]]]
```

Out[∘]= $\frac{e^{s t}\left(b+a\left(-1+e^{b t}\right)\right) f (b+s)}{b (b+a f-b f+s-f s)+e^{s t} f\left(a e^{b t} s-(a-b)(b+s)\right)}$

Inserting *a, b gives*

In[∘]:=
```
ab = {a → (1 + s) (ξ + κ - 1), b → (1 + s) (ξ - 1)};
```

In[∘]:=
```
xmut[t_] = FullSimplify[xmut[t] /. ab]
```

Out[∘]= $\left(e^{s t} f (-1+\xi+s \xi)\left(\kappa-e^{(1+s) t (-1+\xi)}(-1+\kappa+\xi)\right)\right) / \left(\left(-1+e^{s t}\right) f \kappa (-1+\xi)-(-1+\xi)(-1+\xi+s \xi)+f s\left(-1+\kappa+\xi+\left(-1+e^{s t}\right) \kappa \xi-e^{t(-1+\xi+s \xi)}(-1+\kappa+\xi)\right)\right)$

Decay of wild-type carrying cells $x_0(t)$ and convergence of $\frac{x_{wt}(t)}{x_0(t)}$

Define x_{hf}. For large t, $\frac{1}{t}\log x_{wt}(t)$ converges to (1+s) (-1+ξ)

In[∘]:=
```
ξκ = {ξ → (2 n - 3)/(2 n - 1), κ → 1/(2 n - 1)};
xhf[t_] = FullSimplify[(b * e^(b * t))/(a * e^(b * t) + b - a) /. ab]
xwt[t_] = FullSimplify[1 - xmut[t] * (1 - xhf[t])]
Assuming[{s > 0, (1 + s) ξ > 1, ξ > 0, ξ < 1, f > 0, f < 1, κ < 1 - ξ, κ > 0},
 Limit[1/t Log[xwt[t]], t → ∞]]
```

Out[∘]= $\frac{e^{(1+s)\, t\, (-1+\xi)}\, (-1+\xi)}{-\kappa + e^{(1+s)\, t\, (-1+\xi)}\, (-1+\kappa+\xi)}$

Out[∘]= $\left((-1+\xi)\left(-1-\left(-1+e^{t\,(-1+\xi+s\,\xi)}\right) f\,(s\,(-1+\kappa)+\kappa)+\xi+s\,\xi\right)\right) \Big/ \Big((-1+\xi)\,(-1+\xi+s\,\xi)+ f\left(-\left(-1+e^{s\,t}\right)\kappa\,(-1+\xi)+e^{t\,(-1+\xi+s\,\xi)}\, s\,(-1+\kappa+\xi)-s\left(-1+\kappa+\xi+\left(-1+e^{s\,t}\right)\kappa\,\xi\right)\right)\Big)$

Out[∘]= ConditionalExpression[(1 + s) (-1 + ξ), 2 (1 + s) ξ > 2 + s]

The condition 2 (1+s) ξ>2+s must hold to obtain the analytical result (above) with MMA:

For large t, $\frac{x_{wt}(t)}{x_0(t)}$ converges to

In[∘]:=
```
Assuming[{s > 0, (1 + s) ξ > 1, ξ > 0, ξ < 1, f > 0, f < 1, κ < 1 - ξ, κ > 0},
 Limit[(1 - xmut[t] * (1 - xhf[t]))/(1 - xmut[t]), t → ∞]]
```

Out[∘]= $\frac{s\,(-1+\kappa)+\kappa}{(1+s)\,(-1+\kappa+\xi)}$

Convergence

The plot below shows an example for *n=32* that Limit$\left[\frac{1}{t}\right.$ Log[xwt[t]], t → ∞$\left.\right]$ equals (1+s) (-1+ξ) also if (1+s)ξ>1 ⇔ s > $\frac{1}{n-3/2}$ (left vertical line) holds and 2 (1+s) ξ>2+s s > $\frac{2}{n-5/2}$ (right vertical line) is not fulfilled.

In[•]:=
```
nrul = {n → 16};
slist = {0.01, 0.02, 0.03, 0.04, 0.05,
   0.06, 0.07, 0.08, 0.09, 0.1, 0.15, 0.2, 0.25, 0.3}
(1 + slist) * ξ /. ξ → (2 n - 3)/(2 n - 1) /. nrul
2 (1 + slist)/(2 + slist) ξ /. ξ → (2 n - 3)/(2 n - 1) /. nrul

y1 = Table[Limit[
    1/t Log[xwt[t]] /. {ξ → (2 n - 3)/(2 n - 1), κ → 1/(2 n - 1), f → 0.01} /. nrul, t → ∞], {s, slist}]
y2 = Table[(1 + s) (-1 + ξ) /. ξ → (2 n - 3)/(2 n - 1) /. nrul, {s, slist}]
ListPlot[{Transpose[{slist, y1}], Transpose[{slist, y2}]},
 PlotMarkers → "OpenMarkers",
 GridLines → {{2/(n - 5/2) /. nrul, 1/(n - 3/2) /. nrul}, {0, 0}},
 AxesLabel → {s, "Limit[1/t Log[xwt[t]],t→∞]"}, PlotRange → Full,
 PlotLegends → {"Numerical solution", (1 + s) (-1 + ξ)}]
```

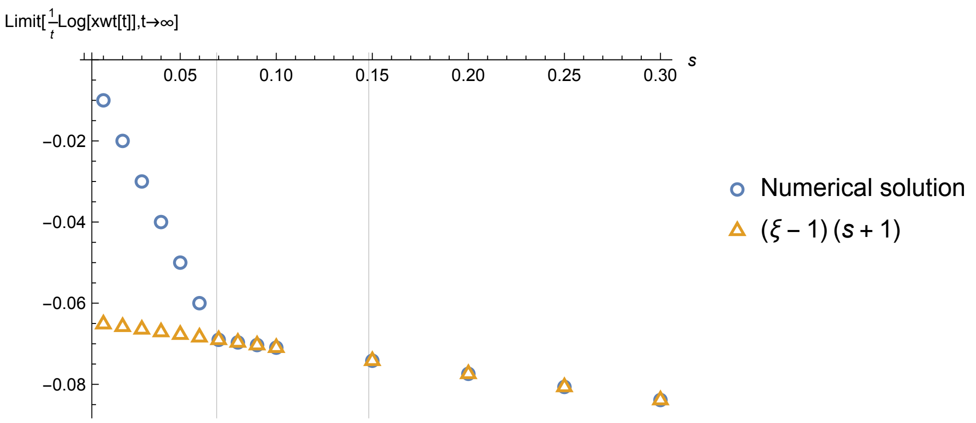
Out[•]= {0.01, 0.02, 0.03, 0.04, 0.05, 0.06, 0.07, 0.08, 0.09, 0.1, 0.15, 0.2, 0.25, 0.3}

Out[•]= {0.944839, 0.954194, 0.963548, 0.972903, 0.982258, 0.991613, 1.00097, 1.01032, 1.01968, 1.02903, 1.07581, 1.12258, 1.16935, 1.21613}

Out[•]= {0.940138, 0.944746, 0.949309, 0.953827, 0.958301, 0.962731, 0.967119, 0.971464, 0.975768, 0.980031, 1.00075, 1.02053, 1.03943, 1.0575}

Out[•]= {-0.01, -0.02, -0.03, -0.04, -0.05, -0.06, -0.0690323, -0.0696774, -0.0703226, -0.0709677, -0.0741935, -0.0774194, -0.0806452, -0.083871}

Out[•]= {-0.0651613, -0.0658065, -0.0664516, -0.0670968, -0.0677419, -0.0683871, -0.0690323, -0.0696774, -0.0703226, -0.0709677, -0.0741935, -0.0774194, -0.0806452, -0.083871}

Out[•]=


Computation of the heterozygosity window

In[∘]:= $\text{Hw}[n\_, s\_] = \text{Simplify}\left[\dfrac{-\text{Log}\left[\frac{s(-1+\kappa)+\kappa}{(1+s)(-1+\kappa+\xi)}\right]}{(1+s)(\xi-1)} \; /. \; \xi \to \dfrac{2n-3}{2n-1} \; /. \; \kappa \to \dfrac{1}{2n-1}\right]$

Out[∘]= $$\frac{(-1+2n)\,\text{Log}\left[\frac{-1+2(-1+n)s}{1+s}\right]}{2(1+s)}$$

In[∘]:= `Plot[Hw[n, 0.3], {n, 1, 100}]`

Out[∘]=