Approximating moments of Wright-Fisher process with a Taylor expansion

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September 9, 2019

1 Framework

Let X_n be a sequence of random variables following a Wright-Fisher bi-allelic allele frequency discrete process. This means that the conditional law of X_{n+1} given X_n is $\frac{1}{N}$ multiplied by a binomial with parameters N and $f(X_n)$ where f is a function, let's say $f \in \mathcal{C}^\infty([0,1],[0,1])$ for simplicity. This is commonly written as

$$X_{n+1}|X_n \sim \frac{1}{N}\mathcal{B}(N, f(X_n)), \quad N \in \mathbb{N}^*, \quad f \in \mathcal{C}^\infty([0,1],[0,1])$$

The aim of this development is to accurately approximate moments of this process without having to compute the whole process transition matrix. One can derive the following identities :

$$\begin{aligned}
E(X_{n+1}) &= E(E(X_{n+1}|X_n)) \\
&= E(f(X_n)) \\
Var(X_{n+1}) &= E(Var(X_{n+1}|X_n)) + Var(E(X_{n+1}|X_n)) \\
&= E\left(\frac{f(X_n)(1-f(X_n))}{N}\right) + Var(f(X_n)) \\
&= \frac{1}{N}E(f(X_n) - \frac{1}{N}E(f^2(X_n)) + Var(f(X_n)) \\
&= \frac{1}{N}E(f(X_n) - \frac{1}{N}\left(Var(X_n) + E(f(X_n))^2\right) + Var(f(X_n)) \\
&= \left(1 - \frac{1}{N}\right)Var(f(X_n)) + \frac{E(f(X_n))(1-E(f(X_n)))}{N}
\end{aligned}$$

As f is non linear function, we can't say that either $E(f(X_n)) = f(E(X_n))$ or $Var(f(X_n)) = f'(E(X_n))Var(X_n)$ (which is the case when f is linear). However we can make an approximation around some point using a Taylor expansion. In the following, μ_n designates a deterministic sequence expected to be our approximation of $E(X_n)$. In the same way, let σ_n^2 designate another deterministic sequence expected to be our approximation of $Var(X_n)$. One can set the error terms for moments : $\varepsilon_n = E(X_n) - \mu_n$ and $\delta_n = Var(X_n) - \sigma_n^2$

One can get the relation :

$$\begin{aligned}
E((X_n - \mu_n)^2) &= E((X_n - E(X_n) + E(X_n) - \mu_n)^2) \\
&= E\left[(X_n - E(X_n))^2 + 2(X_n - E(X_n))(E(X_n) - \mu_n) + (E(X_n) - \mu_n)^2\right] \\
&= Var(X_n) + \varepsilon_n^2 \\
&= \sigma_n^2 + \delta_n + \varepsilon_n^2 \\
&\approx \sigma_n^2.
\end{aligned}$$

2 Lacerda and Seoighe (2014) approximation

Let's do a 1$^{\text{st}}$ order taylor expansion of any function g around the mean of a random variable X :

$$g(X) \approx g(E(X)) + g'(E(X))(X - E(X))$$

So by using mean and variance properties, one gets :

$$\begin{aligned}
E(g(X)) &\approx g(E(X)) \\
Var(g(X)) &\approx g'(E(X))^2 Var(X)
\end{aligned}$$

With these approximations, one get immediately the following identities :

$$\begin{aligned}
E(X_{n+1}) &= E(f(X_n)) \\
&\approx f(E(X_n)) \\
Var(X_{n+1}) &= E\left(\frac{f(X_n)(1-f(X_n))}{N}\right) + Var(f(X_n)) \\
&\approx \frac{f(E(X_n))(1-f(E(X_n)))}{N} + f'(E(X_n))^2 Var(X_n)
\end{aligned}$$

3 Terhorst *et al.* (2015) approximation

Let's write $X_n = x_n + \delta X_n$ where $x_{n+1} = f(x_n)$ and do the 2^{nd} order Taylor expansion about this quantity:

$$\begin{aligned}
E(X_{n+1}) &= E(f(X_n)) \\
&= E(f(x_n + \delta X_n)) \\
&\approx f(x_n) + f'(x_n)E(\delta X_n) + \frac{f''(x_n)}{2}E(\delta X_n^2)
\end{aligned}$$

By definition, one gets also :

$$\begin{aligned}
E(X_{n+1}) &= x_{n+1} + E(\delta X_{n+1}) \\
&= f(x_n) + E(\delta X_{n+1})
\end{aligned}$$

So one can obtain the δX_n recursion:

$$E(\delta X_{n+1}) = f'(x_n)E(\delta X_n) + \frac{f''(x_n)}{2}E((\delta X_n)^2)$$

At this point, the author said :

> Inductively, assuming that we can compute $E(\delta X_n)$ and $E((\delta X_n)^2)$, this enable us to compute $E(X_n)$ and $Var(X_n) = Var(\delta X_n)$. This approach was previously employed by Barton *et al.* (2005) to obtain order $O(\frac{1}{N})$ approximations to these moments. Here, we have used the same idea but automated the symbolic algebra and code generation needed to generate the recursions to higher orders of accuracy.

This suggest that the author did implement an higher order recursion. However, they have not given recursions for $E((\delta X_n)^2)$, so assuming they did in the same way than for $E(\delta X_n)$, one obtains :

$$\begin{aligned}
Var(\delta X_{n+1}) &= Var(X_{n+1}) \\
&= \frac{1}{N}E(f(X_n)(1-f(X_n)) + Var(f(X_n)) \\
&= \frac{1}{N}E(f(x_n+\delta X_n)(1-f(x_n+\delta X_n)) + Var(f(x_n+\delta X_n)) \\
&\approx \frac{1}{N}\left[f(x_n)(1-f(x_n)) + f'(x_n)(1-2f(x_n))E(\delta X_n) - f'(x_n)^2 E(\delta X_n^2)\right] + f'(x_n)^2 Var(\delta X_n)
\end{aligned}$$

Using the fact that $Var(\delta X_n) = E((\delta X_n)^2) - E(\delta X_n)^2$, all relations needed are established

4 Taylor expansion

Remember the following recursions :

$$\begin{aligned}
E(X_{n+1}) &= E(f(X_n)) \\
Var(X_{n+1}) &= \left(1-\frac{1}{N}\right)Var(f(X_n)) + \frac{1}{N}E(f(X_n))(1-E(f(X_n)))
\end{aligned}$$

As f isn't linear, $E(f(X_n))$ has no closed form (the same problem occurs for higher moments). So one needs approximation for these quantities.

To do this, one can expand the f function around our mean approximation μ_n assumed to be close from $E(X_n)$. The following formula could help in that way :

$$f(X_n) = \sum_{k=0}^{+\infty} \frac{f^{(k)}(\mu_n)}{k!}(X_n-\mu_n)^k = f(\mu_n) + f'(\mu_n)(X_n-\mu_n) + \frac{f''(\mu_n)}{2}(X_n-\mu_n)^2 + \dots$$

Assuming that all the operations done are legal, one can optain the following relations :

Proposition 1.

$$E(f(X_n)) = \sum_{k=0}^{+\infty} \frac{f^{(k)}(\mu_n)}{k!} E\left((X_n-\mu_n)^k\right)$$
$$Var(f(X_n) = \sum_{k=2}^{+\infty}\sum_{l=1}^{k-1} \frac{f^{(l)}(\mu_n)f^{(k-l)}(\mu_n)}{l!(k-l)!}\left(E\left((X_n-\mu_n)^k\right) - E\left((X_n-\mu_n)^l\right)E\left((X_n-\mu_n)^{k-l}\right)\right)$$

Proof. By taking the expectancy of $f(X_n) = \sum_{k=0}^{+\infty} \frac{f^{(k)}(\mu_n)}{k!}(X_n-\mu_n)^k$ One gets immediately the 1st point :

$$E(f(X_n)) = E\left(\sum_{k=0}^{+\infty} \frac{f^{(k)}(\mu_n)}{k!}(X_n-\mu_n)^k\right) = \sum_{k=0}^{+\infty} \frac{f^{(k)}(\mu_n)}{k!} E\left((X_n-\mu_n)^k\right)$$

In the same way, the other relations can be obtained :

$$\begin{aligned}
Var(f(X_n)) &= Var\left(\sum_{k=0}^{+\infty} \frac{f^{(k)}(\mu_n)}{k!}(X_n-\mu_n)^k\right) \\
&= E\left(\left(\sum_{k=0}^{+\infty} \frac{f^{(k)}(\mu_n)}{k!}(X_n-\mu_n)^k\right)^2\right) - E\left(\sum_{k=0}^{+\infty} \frac{f^{(k)}(\mu_n)}{k!}(X_n-\mu_n)^k\right)^2 \\
&= E\left(\sum_{k=0}^{+\infty}\left(\sum_{l=0}^{k} \frac{f^{(l)}(\mu_n)f^{(k-l)}(\mu_n)}{l!(k-l)!}\right)(X_n-\mu_n)^k\right) - \left(\sum_{k=0}^{+\infty} \frac{f^{(k)}(\mu_n)}{k!} E\left((X_n-\mu_n)^k\right)\right)^2 \\
&= E\left(\sum_{k=0}^{+\infty}\left(\sum_{l=0}^{k} \frac{f^{(l)}(\mu_n)f^{(k-l)}(\mu_n)}{l!(k-l)!}\right)(X_n-\mu_n)^k\right) \\
&- \sum_{k=0}^{+\infty}\left(\sum_{l=0}^{k} \frac{f^{(l)}(\mu_n)f^{(k-l)}(\mu_n)}{l!(k-l)!} E\left((X_n-\mu_n)^l\right)E\left((X_n-\mu_n)^{k-l}\right)\right) \\
&= \sum_{k=0}^{+\infty}\sum_{l=0}^{k} \frac{f^{(l)}(\mu_n)f^{(k-l)}(\mu_n)}{l!(k-l)!}\Big[E\left((X_n-\mu_n)^k\right) - E\left((X_n-\mu_n)^l\right)E\left((X_n-\mu_n)^{k-l}\right)\Big] \\
&= \sum_{k=2}^{+\infty}\sum_{l=1}^{k-1} \frac{f^{(l)}(\mu_n)f^{(k-l)}(\mu_n)}{l!(k-l)!}\Big[E\left((X_n-\mu_n)^k\right) - E\left((X_n-\mu_n)^l\right)E\left((X_n-\mu_n)^{k-l}\right)\Big] \\
&= f'(\mu_n)^2\left[E((X_n-\mu_n)^2) - E(X_n-\mu_n)^2\right] \\
&+ f'(\mu_n)f''(\mu_n)\left[E((X_n-\mu_n)^3) - E(X_n-\mu_n)E((X_n-\mu_n)^2)\right] \\
&+ \frac{f'(\mu_n)f'''(\mu_n)}{3}\left[E((X_n-\mu_n)^4) - E(X_n-\mu_n)E((X_n-\mu_n)^3)\right] + \frac{f''(\mu_n)^2}{4}\left[E((X_n-\mu_n)^4) - E((X_n-\mu_n)^2)^2\right] \\
&+ \sum_{k=5}^{+\infty}\sum_{l=1}^{k-1} \frac{f^{(l)}(\mu_n)f^{(k-l)}(\mu_n)}{l!(k-l)!}\Big[E\left((X_n-\mu_n)^k\right) - E\left((X_n-\mu_n)^l\right)E\left((X_n-\mu_n)^{k-l}\right)\Big]
\end{aligned}$$

□

4.1 1st order approximation

In this part, the previous relations are used to pinpoint recursions between the moments approximations :

$$
\begin{aligned}
E(X_{n+1}) &= E(f(X_n)) \\
&= \sum_{k=0}^{+\infty} \frac{f^{(k)}(\mu_n)}{k!} E\left((X_n-\mu_n)^k\right) \\
&= f(\mu_n) + f'(\mu_n)\varepsilon_n + \sum_{k=2}^{+\infty} \frac{f^{(k)}(\mu_n)}{k!} E\left((X_n-\mu_n)^k\right)
\end{aligned}
$$

$$
\begin{aligned}
Var(X_{n+1}) &= \left(1-\frac{1}{N}\right) Var(f(X_n)) + \frac{E(f(X_n))(1-E(f(X_n)))}{N} \\
&= \left(1-\frac{1}{N}\right) f'(\mu_n)\left[\sigma_n^2 + \delta_n + \varepsilon_n^2 - \varepsilon_n^2\right] \\
&+ \left(1-\frac{1}{N}\right) \sum_{k=3}^{+\infty}\sum_{l=1}^{k-1} \frac{f^{(l)}(\mu_n) f^{(k-l)}(\mu_n)}{l!(k-l)!} \Big[E\left((X_n-\mu_n)^k\right) - E\left((X_n-\mu_n)^l\right) E\left((X_n-\mu_n)^{k-l}\right)\Big] \\
&+ \frac{1}{N}\left(f(\mu_n) + f'(\mu_n)\varepsilon_n + \sum_{k=2}^{+\infty} \frac{f^{(k)}(\mu_n)}{k!} E\left((X_n-\mu_n)^k\right)\right)\left(1 - f(\mu_n) - f'(\mu_n)\varepsilon_n - \sum_{k=2}^{+\infty} \frac{f^{(k)}(\mu_n)}{k!} E\left((X_n-\mu_n)^k\right)\right) \\
&= \left(1-\frac{1}{N}\right) f'(\mu_n)\sigma_n^2 + \frac{1}{N} f(\mu_n)(1-f(\mu_n)) \\
&+ \left(1-\frac{1}{N}\right) f'(\mu_n)\delta_n + \frac{1}{N} f'(\mu_n)(1-2f(\mu_n))\varepsilon_n - f'(\mu_n)^2\varepsilon_n^2 \\
&+ \sum_{k=2}^{+\infty} \frac{f^{(k)}(\mu_n)}{k!} E\left((X_n-\mu_n)^k\right)\left(1 - 2f(\mu_n) - 2f'(\mu_n)\varepsilon_n - \sum_{k=2}^{+\infty} \frac{f^{(k)}(\mu_n)}{k!} E\left((X_n-\mu_n)^k\right)\right) \\
&+ \left(1-\frac{1}{N}\right) \sum_{k=3}^{+\infty}\sum_{l=1}^{k-1} \frac{f^{(l)}(\mu_n) f^{(k-l)}(\mu_n)}{l!(k-l)!} \Big[E\left((X_n-\mu_n)^k\right) - E\left((X_n-\mu_n)^l\right) E\left((X_n-\mu_n)^{k-l}\right)\Big]
\end{aligned}
$$

4.2 2^nd^ order approximation

In this part, the previous relations are used to pinpoint recursions between the moments approximations :

$$
\begin{aligned}
E(X_{n+1}) &= E(f(X_n)) \\
&= \sum_{k=0}^{+\infty} \frac{f^{(k)}(\mu_n)}{k!} E\left((X_n-\mu_n)^k\right) \\
&= f(\mu_n) + f'(\mu_n)\varepsilon_n + \frac{f''(\mu_n)}{2} E\left((X_n-\mu_n)^2\right) + \sum_{k=3}^{+\infty} \frac{f^{(k)}(\mu_n)}{k!} E\left((X_n-\mu_n)^k\right)
\end{aligned}
$$

$$
\begin{aligned}
Var(X_{n+1}) &= \left(1-\frac{1}{N}\right)Var(f(X_n)) + \frac{E(f(X_n))(1-E(f(X_n)))}{N} \\
&= \left(1-\frac{1}{N}\right)f'(\mu_n)\left[\sigma_n^2+\delta_n+\varepsilon_n^2-\varepsilon_n^2\right] \\
&+ \left(1-\frac{1}{N}\right)\sum_{k=3}^{+\infty}\sum_{l=1}^{k-1}\frac{f^{(l)}(\mu_n)f^{(k-l)}(\mu_n)}{l!(k-l)!}\Big[E\left((X_n-\mu_n)^k\right)-E\left((X_n-\mu_n)^l\right)E\left((X_n-\mu_n)^{k-l}\right)\Big] \\
&+ \frac{1}{N}\Bigg[\left(f(\mu_n)+f'(\mu_n)\varepsilon_n+\frac{f''(\mu_n)}{2}E\left((X_n-\mu_n)^2\right)+\sum_{k=3}^{+\infty}\frac{f^{(k)}(\mu_n)}{k!}E\left((X_n-\mu_n)^k\right)\right) \\
&\times\left(1-f(\mu_n)-f'(\mu_n)\varepsilon_n-\frac{f''(\mu_n)}{2}E\left((X_n-\mu_n)^2\right)-\sum_{k=3}^{+\infty}\frac{f^{(k)}(\mu_n)}{k!}E\left((X_n-\mu_n)^k\right)\right) \\
&= \left(1-\frac{1}{N}\right)f'(\mu_n)\left[\sigma_n^2+\delta_n\right] \\
&+ \left(1-\frac{1}{N}\right)\sum_{k=3}^{+\infty}\sum_{l=1}^{k-1}\frac{f^{(l)}(\mu_n)f^{(k-l)}(\mu_n)}{l!(k-l)!}\Big[E\left((X_n-\mu_n)^k\right)-E\left((X_n-\mu_n)^l\right)E\left((X_n-\mu_n)^{k-l}\right)\Big] \\
&+ \frac{1}{N}\Bigg[\left(f(\mu_n)+f'(\mu_n)\varepsilon_n+\frac{f''(\mu_n)}{2}\left[\sigma_n^2+\delta_n+\varepsilon_n^2\right]+\sum_{k=3}^{+\infty}\frac{f^{(k)}(\mu_n)}{k!}E\left((X_n-\mu_n)^k\right)\right) \\
&\times\left(1-f(\mu_n)-f'(\mu_n)\varepsilon_n-\frac{f''(\mu_n)}{2}\left[\sigma_n^2+\delta_n+\varepsilon_n^2\right]-\sum_{k=3}^{+\infty}\frac{f^{(k)}(\mu_n)}{k!}E\left((X_n-\mu_n)^k\right)\right) \\
&= \left(1-\frac{1}{N}\right)f'(\mu_n)\left[\sigma_n^2+\delta_n\right] \\
&+ \left(1-\frac{1}{N}\right)\sum_{k=3}^{+\infty}\sum_{l=1}^{k-1}\frac{f^{(l)}(\mu_n)f^{(k-l)}(\mu_n)}{l!(k-l)!}\Big[E\left((X_n-\mu_n)^k\right)-E\left((X_n-\mu_n)^l\right)E\left((X_n-\mu_n)^{k-l}\right)\Big] \\
&+ \frac{1}{N}\left(f(\mu_n)+\frac{f''(\mu_n)}{2}\sigma_n^2\right)\left(1-f(\mu_n)-\frac{f''(\mu_n)}{2}\sigma_n^2\right) \\
&+ \frac{1}{N}f'(\mu_n)(1-2f(\mu_n)-f''(\mu_n)\sigma_n^2)\varepsilon_n+\frac{1}{N}\frac{f''(\mu_n)}{2}(1-2f(\mu_n)-f''(\mu_n)\sigma_n^2)\delta_n \\
&+ \frac{1}{N}\left(\frac{f''(\mu_n)}{2}(1-2f(\mu_n)-f''(\mu_n)\sigma_n^2)-f'(\mu_n)^2\right)\varepsilon_n^2 \\
&- \frac{1}{N}\frac{f''(\mu_n)^2}{4}\delta_n^2-\frac{1}{N}f'(\mu_n)f''(\mu_n)\varepsilon_n\delta_n-\frac{1}{N}f'(\mu_n)f''(\mu_n)\varepsilon_n^3-\frac{1}{N}\frac{f''(\mu_n)^2}{2}\delta_n\varepsilon_n^2-\frac{f''(\mu_n)^2}{4}\varepsilon_n^4 \\
&+ \frac{1}{N}\sum_{k=3}^{+\infty}\frac{f^{(k)}(\mu_n)}{k!}E\left((X_n-\mu_n)^k\right)\left(1-2f(\mu_n)-2f'(\mu_n)\varepsilon_n-f''(\mu_n)\left[\sigma_n^2+\delta_n+\varepsilon_n^2\right]-\sum_{k=3}^{+\infty}\frac{f^{(k)}(\mu_n)}{k!}E\left((X_n-\mu_n)^k\right)\right) \\
&= \left(1-\frac{1}{N}\right)f'(\mu_n)\sigma_n^2+\frac{1}{N}\left(f(\mu_n)+\frac{f''(\mu_n)}{2}\sigma_n^2\right)\left(1-f(\mu_n)-\frac{f''(\mu_n)}{2}\sigma_n^2\right) \\
&+ \frac{1}{N}f'(\mu_n)(1-2f(\mu_n)-f''(\mu_n)\sigma_n^2)\varepsilon_n+\frac{1}{N}\frac{f''(\mu_n)}{2}(1-2f(\mu_n)-f''(\mu_n)\sigma_n^2)\delta_n+\left(1-\frac{1}{N}\right)f'(\mu_n)\delta_n \\
&+ \frac{1}{N}\left(\frac{f''(\mu_n)}{2}(1-2f(\mu_n)-f''(\mu_n)\sigma_n^2)-f'(\mu_n)^2\right)\varepsilon_n^2-\frac{1}{N}\frac{f''(\mu_n)^2}{4}\delta_n^2-\frac{1}{N}f'(\mu_n)f''(\mu_n)\varepsilon_n\delta_n \\
&- \frac{1}{N}f'(\mu_n)f''(\mu_n)\varepsilon_n^3-\frac{1}{N}\frac{f''(\mu_n)^2}{2}\delta_n\varepsilon_n^2-\frac{f''(\mu_n)^2}{4}\varepsilon_n^4 \\
&+ \left(1-\frac{1}{N}\right)\sum_{k=3}^{+\infty}\sum_{l=1}^{k-1}\frac{f^{(l)}(\mu_n)f^{(k-l)}(\mu_n)}{l!(k-l)!}\Big[E\left((X_n-\mu_n)^k\right)-E\left((X_n-\mu_n)^l\right)E\left((X_n-\mu_n)^{k-l}\right)\Big] \\
&+ \frac{1}{N}\sum_{k=3}^{+\infty}\frac{f^{(k)}(\mu_n)}{k!}E\left((X_n-\mu_n)^k\right)\left(1-2f(\mu_n)-2f'(\mu_n)\varepsilon_n-f''(\mu_n)\left[\sigma_n^2+\delta_n+\varepsilon_n^2\right]-\sum_{k=3}^{+\infty}\frac{f^{(k)}(\mu_n)}{k!}E\left((X_n-\mu_n)^k\right)\right)
\end{aligned}
$$

5 Recursions

With the relations established before, one can set different recursions schemes to approximate moments of the true Wright-Fisher process.

5.1 From Lacerda and Seoighe's derivations

One obtained the following :

$$E(X_{n+1}) \approx f(X_n) \tag{1}$$

$$Var(X_{n+1}) \approx \frac{1}{N} f(E(X_n))(1 - f(E(X_n))) + f'(E(X_n))^2 Var(X_n) \tag{2}$$

Recursions used by Lacerda and Seoighe are:

$$\begin{aligned}
\mu_0 &= E(X_0) \\
\sigma_0^2 &= Var(X_0) \\
\mu_{n+1} &= f(\mu_n) \\
\sigma_{n+1}^2 &= \frac{f(\mu_n)(1 - f(\mu_n))}{N} + f'(\mu_n)^2 \sigma_n^2
\end{aligned}$$

Remark 1. *Note that in their case, they took f having the form :*

$$f(x) = \frac{(1+s)x(1-\alpha) + (1-x)\beta}{1+sx}$$

5.2 From Terhorst *et al.*'s (2015) derivations

These recursions are more tricky than previous ones because it's a five crossed sequence recursion :

$$\begin{aligned}
x_0 = \mu_0 &= E(X_0) & \qquad \varepsilon_{1,0} &= 0 \\
\sigma_0^2 &= Var(X_0) & \qquad \varepsilon_{2,0} &= 0
\end{aligned}$$

$$\begin{aligned}
x_{n+1} &= f(x_n) \\
\mu_{n+1} &= f(x_n) + f'(x_n)\varepsilon_{1,n} + \frac{f''(x_n)}{2}\varepsilon_{2,n} \\
\sigma_{n+1}^2 &= \frac{1}{N}\left[f(x_n)(1 - f(x_n)) + f'(x_n)(1 - 2f(x_n))\varepsilon_{1,n} - f'(x_n)^2\varepsilon_{2,n}\right] + f'(x_n)^2\sigma_n^2 \\
\varepsilon_{1,n+1} &= f'(x_n)\varepsilon_{1,n} + \frac{f''(x_n)}{2}\varepsilon_{2,n} \\
\varepsilon_{2,n+1} &= \sigma_{n+1}^2 + \varepsilon_{1,n+1}^2
\end{aligned}$$

5.3 From 1$^{\text{st}}$ order approximation

In this part, let μ_n, σ_n^2 be defined from the following relations :

$$\begin{aligned}
\mu_0 &= E(X_0) \\
\sigma_0^2 &= Var(X_0) \\
\mu_{n+1} &= f(\mu_n) \\
\sigma_{n+1}^2 &= \frac{f(\mu_n)(1 - f(\mu_n))}{N} + \left(1 - \frac{1}{N}\right) f'(\mu_n)^2 \sigma_n^2
\end{aligned}$$

5.4 From 2$^{\text{nd}}$ order approximation

In this part, let μ_n, σ_n^2 be defined from the following relations :

$$\begin{aligned}
\mu_0 &= E(X_0) \\
\sigma_0^2 &= Var(X_0) \\
\mu_{n+1} &= f(\mu_n) + \frac{f''(\mu_n)}{2}\sigma_n^2 \\
\sigma_{n+1}^2 &= \frac{1}{N}\left(f(\mu_n) + \frac{f''(\mu_n)}{2}\sigma_n^2\right)\left(1 - f(\mu_n) - \frac{f''(\mu_n)}{2}\sigma_n^2\right) + \left(1 - \frac{1}{N}\right) f'(\mu_n)^2 \sigma_n^2
\end{aligned}$$